4 The Real Numbers

An axiomatic treatment of the real numbers provides a firm basis for our reasoning, and it gives us a framework for studying some subtle questions concerning irrational numbers.

To such questions as, "how do we know that there is a number whose square is 2?" and "how is π constructed?" it is tempting to give geometric answers: "$\sqrt{2}$ is the length of the diagonal of square whose side has length 1"; "π is the ratio of the circumference of a circle to its diameter." Answers like this are unsatisfactory, though, since they rely too much on intuition. How does one *define* the circumference of a circle? These questions have been of serious concern to mathematicians for centuries. One way to settle them, without recourse to geometric intuition, is to write down a list of unambiguous rules or *axioms* which enable us to prove all we want.

A set of axioms for the real numbers was developed in the middle part of the 19th Century. These particular axioms have proven their worth without much doubt,* and we will take them for our starting point. In more advanced courses one has to face the question of showing that there exists a system of numbers obeying these axioms, but we shall merely assume this here.

Mathematics that is useful in applications to science is rarely discovered by means of axiom systems. Axiomatics is more frequently the final product of a piece of mathematics created for some need. The axioms for real numbers were agreed on only after centuries of trial and error, and only after the basic theorems were already discovered.

It is *not* our intent to show that *all* the usual manipulative rules follow from the axioms, since that job is too long and is done in algebra courses. Our aim is merely to set out our assumptions in a clear fashion and to give a few illustrations of how to use them.

Addition and Multiplication Axioms

Our first axioms pertain to the operation of addition.

*There is still some controversy remaining. Some mathematicians prefer a "constructive" or "intuitionistic" approach; see Heyting, *Intuitionism, An Introduction*, North-Holland (1956), or Bishop, *Foundations of Constructive Analysis*, McGraw-Hill (1967).

I. Addition Axioms There is an addition operation "+" which associates to every two real numbers x and y a real number $x + y$ called the *sum* of x and y such that:

1. For all x and $y, x + y = y + x$ [commutativity].
2. For all x, y, and $z, x + (y + z) = (x + y) + z$ [associativity].
3. There is a number 0 ("zero") such that, for all x, $x + 0 = x$ [existence of additive identity].
4. For each x, there is a number $-x$ such that $x + (-x) = 0$ [existence of additive inverses].

On the basis of addition axiom (4), we can define the operation of subtraction by $x - y = x + (-y)$.

The next axioms pertain to multiplication and its relation with addition.

II. Multiplication Axioms There is a multiplication operation "$\cdot$" which associates to every two real numbers x and y a real number $x \cdot y$, called the *product* of x and y, such that:

1. For all x and $y, x \cdot y = y \cdot x$ [commutativity].
2. For all x, y, and $z, x \cdot (y \cdot z) = (x \cdot y) \cdot z$ [associativity].
3. There is a number 1, which is different from 0, such that, for all $x, x \cdot 1 = x$ [existence of multiplicative identity].
4. For each $x \neq 0$, there is a number $1/x$ such that

   $$\left(\frac{1}{x} \cdot x\right) = 1$$

   [existence of multiplicative inverses].
5. For all x, y, and $z, x \cdot (y + z) = x \cdot y + x \cdot z$ [distributivity].

On the basis of multiplication axiom 4, we can define the operation of division by

$$\frac{x}{y} = x \cdot \left(\frac{1}{y}\right) \quad (\text{if } y \neq 0)$$

One often writes xy for $x \cdot y$.

Implicit in our rules for addition and multiplication is that the numbers $x + y$ and xy are uniquely specified once x and y are given. Thus we have, for example, the usual rule of algebraic manipulation, "if $x = z$, then $xy = zy$." In fact, $x = z$ means that x and z are the same number, so that multiplying each of them by y must give the same result.

In principle, one could prove all the usual rules of algebraic manipulation from the axioms above, but we will content ourselves with the few samples given in the exercises below.

A final remark: we can define $2 = 1 + 1, 3 = 2 + 1, \ldots$, and via division obtain the fractions. As usual, we write x^2 for $x \cdot x$, x^3 for $x^2 \cdot x$, etc.

Solved Exercises*

1. Prove that $x \cdot 0 = 0$ by multiplying the equality $0 + 0 = 0$ by x.
2. Prove that $(x + y)^2 = x^2 + 2xy + y^2$.
3. Prove that $2 \cdot 3 = 6$.
4. Prove that $(-x) \cdot y = -(xy)$.
5. Prove that $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ if $b \neq 0$ and $d \neq 0$.

Exercises

Prove the following identities.

1. $(-x)(-y) = xy$
2. $(x - y)^2 = x^2 - 2xy + y^2$
3. $(x + y)(x - y) = x^2 - y^2$
4. $\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{ac}{bd} \quad (b \neq 0, d \neq 0)$
5. $\frac{1}{(a/b)} = \frac{b}{a} \quad (a \neq 0, b \neq 0)$
6. $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{a}{b} + \frac{b}{a} + 2 \quad (a \neq 0, b \neq 0)$
7. $(-x)\left(\frac{1}{x}\right) = -1 \quad (x \neq 0)$

*Solutions appear in the Appendix.

8. $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$
9. $a+1 \neq a$
10. If $a+b=a+c$, then $b=c$.

Order Axioms

From now on we will use, without further justification, the usual rules for algebraic manipulations. The previous exercises were intended to convince the reader that these rules can all be derived from the addition and multiplication axioms. We turn now to the order axioms.

III. Order Axioms There is a relation "$\leqslant$" such that, for certain pairs x and y of real numbers the statement "$x \leqslant y$" (read "x is less than or equal to y") is true. This relation has the following properties:

1. If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$ [transitivity].
2. If $x = y$, then $x \leqslant y$ [reflexivity].
3. If $x \leqslant y$ and $y \leqslant x$, then $x = y$ [asymmetry].
4. For any numbers x and y, either $x \leqslant y$ or $y \leqslant x$ is true [comparability].
5. If $x \leqslant y$, and z is any number, then $x+z \leqslant y+z$.
6. If $0 \leqslant x$ and $0 \leqslant y$, then $0 \leqslant xy$.

We write $x < y$ (x is strictly less than y) if $x \leqslant y$ and $x \neq y$. Also, we write $y \geqslant x$ (y is greater than or equal to x) if $x \leqslant y$, and $y > x$ (y is strictly greater than x) if $x < y$. Again, one can prove all the usual properties of the inequality signs from the axioms above. As before, we limit ourselves to a few instances.

Solved Exercises

6. Prove that $0 < 1$.
7. Prove: if $x \leqslant y$ and $c \leqslant 0$, then $cx \geqslant cy$.
8. Let a and b be numbers such that, for any number c with $c < a$, we must have $c < b$. Prove that $a \leqslant b$.
9. Prove: $x^2 \geqslant 0$ for all x.

Exercises

Prove the following statements.

11. $1 < 2$
12. If $a > 1$, then $1/a < 1$.
13. If $a^2 < a$, then $0 < a < 1$.
14. If $a > 0$ and $b < 0$, then $(a + b)^2 < (a - b)^2$.
15. If a and b have the same sign, then $(a + b)^2 > (a - b)^2$.
16. If $x < y$, and z is any number, then $x - z < y - z$.
17. If $a < b$ and $c < d$, then $a + c < b + d$.
18. If a is less than every positive real number, then $a \leqslant 0$.
19. A number a was called *infinitesimal* by the founders of calculus if: (1) $a > 0$; and (2) a is less than every positive real number. Prove that there are no infinitesimal real numbers.*

The Completeness Axiom

We need one more axiom to guarantee that irrational numbers exist. In fact, the rational numbers (i.e., quotients of integers) satisfy all of the axioms in I, II, and III.

To motivate the last axiom, we consider the problem of defining $\sqrt{2}$. Consider the set S consisting of all numbers such that $0 \leqslant x$ and $x^2 \leqslant 2$. If x_1 and x_2 are elements of S and y is a number between them, i.e., $x_1 < y < x_2$, then the order axioms imply that $0 \leqslant y$ (since $0 \leqslant x_1$) and also that $y^2 \leqslant 2$ (since $0 \leqslant y \leqslant x_2$), so that y is again in S. Thus, the set S has "no holes" in the sense that it contains every number between any two of its members. Our intuition tells us that S ought to be an interval. If we could be sure that this were so, we could look at the right-hand endpoint of S (which cannot be ∞, since $1 \in S$ but $2 \notin S$) and establish that $c^2 = 2$. (This is done in Solved Exercise 11.) We would then have found the square root of 2. The completeness axiom makes our intuitive notion into a property of real numbers, taking its place alongside the addition, multiplication, and order axioms. We need one definition before stating the axiom.

*There is a modern theory of infinitesimals, but they are not real numbers. The theory is called "nonstandard analysis." (See Keisler, H., *Elementary Calculus:* Prindle, Weber, and Schmidt, Boston (1976).)

Definition A set S of real numbers is *convex* if, whenever x_1 and x_2 belong to S and y is a number such that $x_1 < y < x_2$, then y belongs to S as well.

Any interval is a convex set. (See Solved Exercise 10.) The completeness axiom asserts the converse.

IV. Completeness Axiom Every convex set of real numbers is an interval.

The force of the completeness axiom lies in the fact that intervals have endpoints. Thus, whenever we can prove a set to be convex, the completeness axiom implies the existence of certain real numbers.

Here is some further motivation for the completeness axiom. Suppose that S is a convex set of real numbers which does not extend infinitely in either direction on the number line. Imagine placing the tips of a pair of calipers on two points of S. (See Fig. 4-1.) If x_1 and x_2 are not endpoints of S, we can imagine spreading the calipers to rest on points y_1 and y_2 in S. If y_1 and y_2 are still not endpoints, we can imagine spreading the calipers more and more until no more spreading is possible. The points beyond which the caliper tips cannot spread must be endpoints of S; they may or may not belong to S.

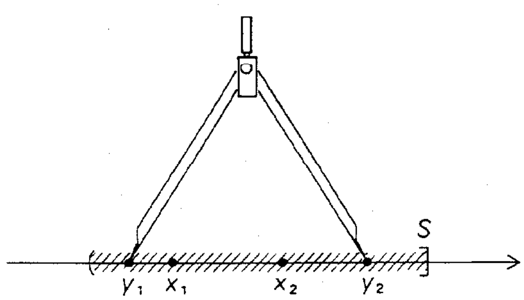

Fig. 4-1 The completeness axiom illustrated by the spreading calipers.

Solved Exercises

10. Prove that $[a, b)$ is convex.

11. Let S be the set consisting of those x for which $0 \leqslant x$ and $x^2 \leqslant 2$. Let c be the right-hand endpoint of S (which exists by the completeness axiom). Prove that $c^2 = 2$; i.e., prove the existence of $\sqrt{2}$. [*Hint*: Show that if $c^2 < 2$, $0 < h < 1$, and $0 < h < (2 - c^2)/(2c + 1)$, then $(c + h)^2 < 2$.]

12. Prove that $\sqrt{2}$ is irrational; i.e., show that there is no rational number m/n such that $(m/n)^2 = 2$. [*Hint*: Suppose that m and n have no common factor; is m even or odd?]

13. Prove that any open interval (a, b) contains both rational and irrational numbers.

Exercises

20. Let S be the set consisting of those numbers x for which $x \in [0,1)$ or $x \in (1,2]$. Prove that S is not convex.

21. Prove that (a, ∞) and $(-\infty, b]$ are convex.

22. For which values of a, b, and c is the set of all x such that $ax^2 + bx + c < 0$ convex? What are the endpoints of the convex set?

23. Prove that, if S and T are convex sets, then the set $S + T$, consisting of all sums $x + y$ with $x \in S$ and $y \in T$, is convex. How are the endpoints of S and T related to the endpoints of $S + T$? If S is open and T is closed, is $S + T$ open or closed?

24. Let A and B be sets of real numbers such that every element of A is less than every element of B, and such that every real number belongs to either A or B. Using the completeness axiom, show that there is exactly one real number c such that every number less than c is in A and every number greater than c is in B.

Problems for Chapter 4

1. Using the addition and multiplication axioms as stated, prove the following identities.
   (a) $(x + y) + (z + w) = (x + (y + z)) + w$
   (b) $x(a + (b + c)) = xa + (xb + xc)$

2. Prove the following identities.
   (a) $a - (-b) = a + b$
   (b) $(x + y)(u + v) = xu + yu + xv + yv$
   (c) $(ab)^2 = a^2b^2$
   (d) $\dfrac{1 - x^3}{1 - x} = 1 + x + x^2 \quad (x \neq 1)$

3. Prove the following statements.
   (a) If $x < 0$, then $x^3 < 0$; if $x > 0$, then $x^3 > 0$.
   (b) x and y have the same sign if and only if $xy > 0$.

(c) If $a \leqslant b$, $b \leqslant c$, and $c \leqslant a$, then $a = b = c$.

(d) If $0 < a < b$, then $0 < 1/b < 1/a$.

4. Let $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3]$, ... be an infinite sequence of closed intervals, each of which is contained in the previous one; i.e., $a_i \leqslant a_{i+1}$, $b_{i+1} \leqslant b_i$.

   (a) Using the completeness axiom, prove that there is a real number c which belongs to *all* the intervals in the sequence. [*Hint*: Consider the set consisting of those x such that $x \leqslant b_n$ for all n.]

   (b) Give a condition on the intervals which will insure that there is exactly one real number which belongs to all the intervals.

   (c) Show that the result in (a) is false if the intervals are open.

   (d) Show that the result in (a) is false if we are working with rational numbers rather than with real numbers.

5. Prove that every real number is less than some positive integer. This result is often referred to as the "Archimedian property." [*Hint*: Consider the set S consisting of those x which are less than some positive integer, and show that S cannot have a finite endpoint.]

6. Which of the following sets are convex?

   (a) All x such that $x^3 < 0$.

   (b) All x such that $x^3 < x$.

   (c) All x such that $x^3 < 4$.

   (d) All the areas of polygons inscribed in the circle $x^2 + y^2 = 1$.

   (e) All x such that the decimal expansion of x begins with 2.95.

7. Prove that $(a + (1/a)) \geqslant 2$ if $a > 0$.

8. Use the axioms for addition and multiplication to prove the following:

   (a) $3 - \frac{1}{2} = 4 - \frac{3}{2}$ (b) $\frac{21}{3} = 6 + 1$

   (c) $4 \cdot 2 + 5 \cdot 6 \neq 31$ (d) $2^3 = 8$

   (e) $-(-(-a)) + a = 0$ (f) $-(a + b) = -a - b$

   (g) $\dfrac{a}{b} - \dfrac{c}{b} = \dfrac{a-c}{b}$, $b \neq 0$ (h) $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$; $b \neq 0$

9. Let S be the set of x such that $x^3 < 10$. Show that S is convex and describe S as an interval. Discuss how this can be used to prove the existence of $\sqrt[3]{10}$.

10. Prove the following inequality using the order axioms:

$$\frac{a^2 + b^2}{2} \geqslant \left(\frac{a+b}{2}\right)^2, \quad \text{where } a < b \text{ and } b < 0.$$

11. Prove that $\sqrt{2} < \sqrt{3}$. [*Hint*: Assume $\sqrt{2} \geqslant \sqrt{3}$ and derive a contradiction.]

12. If $a > 0$, prove that there is a positive integer n such that $0 < 1/n < a$.

13. Suppose that A and B are convex sets such that every element of A is less than every element of B. Show that, if for every positive number ϵ there are

elements a in A and b in B with $b - a < \epsilon$, then there is a transition point from A to B. [*Hint*: Use the completeness axiom to show that A has a right-hand endpoint and B has a left-hand endpoint; then show that these endpoints are equal.]

14. Look up the "least upper bound" version of the completeness axiom* and prove that it is equivalent to ours.

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*See, for example, Spivak, M. *Calculus*, Publish or Perish, Inc. (1980).